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Quasi-periodic solutions of the modified Kadomtsev–Petviashvili equation

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Abstract. The $(2+1)$-dimensional modified Kadomtsev–Petviashvili equation is decomposed into systems of integrable ordinary differential equations resorting to the nonlinearization of Lax pairs. Abel–Jacobi coordinates are introduced to straighten the flows, from which quasi-periodic solutions of the modified Kadomtsev–Petviashvili equation are obtained in terms of Riemann theta functions.

1. Introduction

There are several systematic approaches to obtain explicit solutions of the soliton equations, such as the inverse scattering transformation, the algebro-geometric method, the polar expansion method and others (see, e.g., [1–4] and references therein). Some interesting explicit solutions have been found, the most important among which are pure-soliton solutions, quasi-periodic solutions and polar expansion solutions. Recently two interesting approaches, nonlinearization of Lax pairs [5, 6] and constrained flow [7, 8], have been developed to engender new finite-dimensional completely integrable systems from known soliton hierarchies. Very recently, the nonlinearization approach has been applied successfully to obtaining quasi-periodic solutions of soliton equations [9, 10] and generalized to the investigation of soliton equations in two spatial and one temporal (i.e. $(2+1)$) dimensions.

Consider the $(2+1)$-dimensional modified Kadomtsev–Petviashvili (mKP) equation [11]

$$4q_t = q_{xxx} - 6q^2 q_x - 6q_x \partial_x^{-1} q_y + 3\partial_x^{-1} q_{yy} \tag{1.1}$$

which is the compatibility condition of the following linear system [11, 12]:

$$\begin{aligned} u_y &= u_{xx} - 2qu_x \\ u_t &= u_{xxx} - 3qu_{xx} + \tfrac{3}{2}(q^2 - q_x - \partial_x^{-1} q_y)u_x \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} v_y &= -v_{xx} - 2qv_x \\ v_t &= v_{xxx} + 3qv_{xx} + \tfrac{3}{2}(q^2 + q_x - \partial_x^{-1} q_y)v_x. \end{aligned} \tag{1.3}$$

If we impose the constraint as follows:

$$q = uv \tag{1.4}$$

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then (1.2) and (1.3) are nonlinearized into the generalized nonlinear Schrödinger equation with derivative coupling [13]

$$u_y = u_{xx} - 2uvu_x \qquad v_y = -v_{xx} - 2uvv_x \tag{1.5}$$

and its higher-order equation

$$\begin{aligned} u_t &= u_{xxx} - 3(uvu_{xx} + u_x^2 v - u^2 v^2 u_x) \\ v_t &= v_{xxx} + 3(uvv_{xx} + v_x^2 u + u^2 v^2 v_x). \end{aligned} \tag{1.6}$$

Therefore, if u is a solution of (1.5) and (1.6), q determined by (1.4) is a solution of the mKP equation (1.1). This fact could also be verified by direct calculations.

In this paper, based on the above study we would like to develop further the methods in [14–16] to construct quasi-periodic solutions of the mKP equation (1.1). We first derive the soliton hierarchy associated with (1.5) and (1.6). Secondly, equations (1.5) and (1.6) are decomposed into systems of integrable ordinary differential equations. Finally, a hyperelliptic Riemann surface of genus N and Abel–Jacobi coordinates are introduced to straighten the associated flows, from which the quasi-periodic solutions of the mKP equation (1.1) are expressed explicitly by the Riemann theta functions.

2. The hierarchy

To derive the hierarchy and its stationary hierarchy associated with equations (1.5) and (1.6), we introduce the Lenard gradient sequence S_j, $-1 \leqslant j \in \mathbb{Z}$ by the recursion relation

$$KS_{j-1} = JS_j \qquad S_j|_{(u,v)=0} = 0 \qquad S_{-1} = \left(v, u, \tfrac{1}{2}\right)^T \qquad j \geqslant 0 \tag{2.1}$$

with two operators ($\partial = \partial/\partial x$)

$$K = \begin{pmatrix} 0 & \partial + uv & 0 \\ \partial - uv & 0 & 0 \\ -u & v & \partial \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 & -2u \\ -1 & 0 & 2v \\ -u & v & \partial \end{pmatrix}.$$

It is easy to see that (2.1) implies the relation

$$-uS_j^{(1)} + vS_j^{(2)} + S_{jx}^{(3)} = 0 \tag{2.2}$$

and S_j is uniquely determined by the recursion relation (2.1). Here the condition $S_j|_{(u,v)=0} = 0$ is used only in (2.1) to select constants of integration to be zero. A direct calculation gives from the recursion relation (2.1) that

$$S_0 = \begin{pmatrix} -v_x - uv^2 \\ u_x - u^2 v \\ -uv \end{pmatrix} \qquad S_1 = \begin{pmatrix} v_{xx} - v^2 u_x + 3uvv_x + u^2 v^3 \\ u_{xx} + u^2 v_x - 3uvu_x + u^3 v^2 \\ uv_x - u_x v + u^2 v^2 \end{pmatrix}. \tag{2.3}$$

Consider the spectral problem

$$\varphi_x = U\varphi \qquad U = \begin{pmatrix} \frac{1}{2}(\lambda - uv) & \lambda u \\ v & -\frac{1}{2}(\lambda - uv) \end{pmatrix} \tag{2.4}$$

and the auxiliary problem

$$\varphi_{t_m} = V^{(m)}\varphi \qquad V^{(m)} = \begin{pmatrix} V_{11}^{(m)} & V_{12}^{(m)} \\ V_{21}^{(m)} & -V_{11}^{(m)} \end{pmatrix} \tag{2.5}$$

where

$$V_{11}^{(m)} = a_m + \sum_{j=0}^{m} S_{j-1}^{(3)} \lambda^{m+1-j} \qquad V_{12}^{(m)} = \sum_{j=0}^{m} S_{j-1}^{(2)} \lambda^{m+1-j} \qquad V_{21}^{(m)} = \sum_{j=0}^{m} S_{j-1}^{(1)} \lambda^{m-j}.$$

Then the compatibility condition between (2.4) and (2.5) yields the zero-curvature equation $U_{t_m} - V_x^{(m)} + [U, V^{(m)}] = 0$, which is equivalent to the following soliton equations:

$$\begin{aligned}(uv)_{t_m} &= -2a_{m,x}\\ u_{t_m} &= S^{(2)}_{m-1,x} + uvS^{(2)}_{m-1} + 2ua_m \\ v_{t_m} &= S^{(1)}_{m-1,x} - uvS^{(1)}_{m-1} - 2va_m.\end{aligned} \tag{2.6}$$

Equations (2.6) imply

$$\begin{aligned}a_m &= -\tfrac{1}{2}\partial^{-1}[v\partial S^{(2)}_{m-1} + u\partial S^{(1)}_{m-1} + uv(vS^{(2)}_{m-1} - uS^{(1)}_{m-1})\\ &= \tfrac{1}{2}\partial^{-1}(u\partial S^{(1)}_m - vS^{(2)}_m) = \tfrac{1}{2}S^{(3)}_m\end{aligned} \tag{2.7}$$

where $\partial\partial^{-1} = \partial^{-1}\partial = 1$. By using (2.7) and (2.1), equations (2.6) can be written as

$$(u_{t_m}, v_{t_m})^T = X_m \qquad m \geqslant 0 \tag{2.8}$$

and

$$X_j = \begin{pmatrix} S^{(2)}_j - uS^{(3)}_j \\ -S^{(1)}_j + vS^{(3)}_j \end{pmatrix} = \begin{pmatrix} -u\partial^{-1}u & 1 + u\partial^{-1}v \\ -1 + v\partial^{-1}u & -v\partial^{-1}v \end{pmatrix}\begin{pmatrix} S^{(1)}_j \\ S^{(2)}_j \end{pmatrix}.$$

The first two nontrivial equations in the hierarchy (2.8) with $t_1 = y, t_2 = t$ are exactly equations (1.5) and (1.6).

Assume that (2.4) and (2.5) have two basic solutions $\psi = (\psi_1, \psi_2)^T$ and $\phi = (\phi_1, \phi_2)^T$. We introduce a matrix W of three functions f, g, h by

$$W = \tfrac{1}{2}(\phi\psi^T + \psi\phi^T)\sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix} \qquad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.9}$$

A direct calculation shows that

$$W_x = [U, W] \qquad W_{t_m} = [V^{(m)}, W] \tag{2.10}$$

which implies that the function det W is a constant independent of x and t_m. Equation (2.10) can be written as

$$f_x = \lambda uh - vg \qquad g_x = (\lambda - uv)g - 2\lambda uf \qquad h_x = 2vf - (\lambda - uv)h \tag{2.11}$$

and

$$f_{t_m} = hV^{(m)}_{12} - gV^{(m)}_{21} \qquad g_{t_m} = 2gV^{(m)}_{11} - 2fV^{(m)}_{12} \qquad h_{t_m} = 2fV^{(m)}_{21} - 2hV^{(m)}_{11}. \tag{2.12}$$

Now we suppose that the functions f, g and h are finite-order polynomials in λ:

$$f = \sum_{j=0}^{N} f_{j-1}\lambda^{N+1-j} \qquad g = \sum_{j=0}^{N} g_{j-1}\lambda^{N+1-j} \qquad h = \sum_{j=0}^{N} h_{j-1}\lambda^{N-j}. \tag{2.13}$$

Substituting (2.13) into (2.11) yields

$$KG_{j-1} = JG_j \qquad JG_{-1} = 0 \tag{2.14}$$

$$KG_{N-1} = 0 \qquad G_j = (h_j, g_j, f_j)^T. \tag{2.15}$$

It is easy to see that (2.14) implies

$$-uh_j + vg_j + f_{jx} = 0 \tag{2.16}$$

and the equation $JG_{-1} = 0$ has the general solution

$$G_{-1} = \alpha_0 S_{-1} \tag{2.17}$$

where α_0 is an integral constant. Therefore, if we take (2.17) as a starting point, then G_j can be determined recursively by the relation (2.14). In fact, noticing $\ker J = \{cS_{-1}|\forall c\}$ and acting with the operator $(J^{-1}K)^{k+1}$ upon (2.17), we obtain from (2.14) and (2.1) that

$$G_k = \sum_{j=0}^{k+1} \alpha_j S_{k-j} \qquad -1 \leqslant k \leqslant N-1 \tag{2.18}$$

where $\alpha_0, \ldots, \alpha_{k+1}$ are integral constants. Substituting (2.18) into the first expression of (2.15) yields a certain stationary evolution equation,

$$\alpha_0 \hat{X}_N + \alpha_1 \hat{X}_{N-1} + \cdots + a_N \hat{X}_0 = 0 \tag{2.19}$$

where

$$\hat{X}_j = \begin{pmatrix} 0 & \partial + uv \\ \partial - uv & 0 \end{pmatrix} \begin{pmatrix} S_{j-1}^{(1)} \\ S_{j-1}^{(2)} \end{pmatrix}.$$

This means that expressions (2.13) are existent.

3. Integrable ordinary differential equations

In this section, equations (1.5) and (1.6) will be decomposed into systems of integrable ordinary differential equations. Without any loss of generality we can set $\alpha_0 = 1$, since changing G_{-1} simple results in multiplying f, g and h by a constant. From (2.3), (2.1) and (2.18), we have

$$\begin{aligned}
&f_{-1} = \tfrac{1}{2} \qquad g_{-1} = u \qquad h_{-1} = v \\
&f_0 = -uv + \tfrac{1}{2}\alpha_1 \qquad g_0 = u_x - u^2 v + \alpha_1 u \qquad h_0 = -v_x - uv^2 + \alpha_1 v \\
&f_1 = uv_x - u_x v + u^2 v^2 - \alpha_1 uv + \tfrac{1}{2}\alpha_2 \\
&g_1 = u_{xx} + u^2 v_x - 3uvu_x + u^3 v^2 + \alpha_1 (u_x - u^2 v) + \alpha_2 u \\
&h_1 = v_{xx} - v^2 u_x + 3uvv_x + u^2 v^3 - \alpha_1 (v_x + uv^2) + \alpha_2 v.
\end{aligned} \tag{3.1}$$

By using (2.13), we write g and h as two finite products, they take the form

$$g = \lambda u \prod_{i=1}^{N} (\lambda - \mu_i) \qquad h = v \prod_{i=1}^{N} (\lambda - \nu_i) \tag{3.2}$$

which imply, by comparing the coefficients of the same power for λ, that

$$g_0 = -u \sum_{j=1}^{N} \mu_j \qquad h_0 = -v \sum_{j=1}^{N} \nu_j \tag{3.3}$$

$$g_1 = u \sum_{i<j} \mu_i \mu_j \qquad h_1 = v \sum_{i<j} \nu_i \nu_j. \tag{3.4}$$

Thus from (3.1), (3.3), (3.4) and (1.5), after a simple calculation, we obtain

$$\begin{aligned}
&\partial \ln u - uv + \alpha_1 = -\sum_{j=1}^{N} \mu_j \\
&\partial \ln v + uv - \alpha_1 = \sum_{j=1}^{N} \nu_j
\end{aligned} \tag{3.5}$$

and

$$\partial_y \ln u + uv(\partial \ln v - \partial \ln u + uv) = \sum_{i<j} \mu_i \mu_j + \alpha_1 \sum_{j=1}^{N} \mu_j + \alpha_1^2 - \alpha_2$$
$$\partial_y \ln v - uv(\partial \ln v - \partial \ln u + uv) = -\sum_{i<j} \nu_i \nu_j - \alpha_1 \sum_{j=1}^{N} \nu_j - \alpha_1^2 + \alpha_2 \tag{3.6}$$

which imply

$$\partial \ln uv = \sum_{j=1}^{N} (\nu_j - \mu_j)$$
$$\partial_y \ln uv = \sum_{i<j} (\mu_i \mu_j - \nu_i \nu_j) + \alpha_1 \sum_{j=1}^{N} (\mu_j - \nu_j). \tag{3.7}$$

Resorting to (1.5), (3.5) and the first expression of (3.7), we have

$$\begin{aligned}\partial_y \ln uv &= \frac{u_{xx}}{u} - \frac{v_{xx}}{v} - 2uv\partial \ln uv \\ &= \partial_x^2 \ln \frac{u}{v} + (\partial \ln u)^2 - (\partial \ln v)^2 - 2uv\partial \ln uv \\ &= \left(\sum_{j=1}^{N} \mu_j\right)^2 - \left(\sum_{j=1}^{N} \nu_j\right)^2 - \partial \sum_{j=1}^{N} (\mu_j + \nu_j) + 2(\alpha_1 - uv) \sum_{j=1}^{N} (\mu_j - \nu_j)\end{aligned} \tag{3.8}$$

which together with (1.4), the second expression of (3.7), gives

$$q = \tfrac{1}{2}\alpha_1 + \tfrac{1}{4} \sum_{j=1}^{N} (\mu_j + \nu_j) + \frac{\sum_{j=1}^{N} (\nu_j^2 - \mu_j^2) + 2\partial \sum_{j=1}^{N} (\mu_j + \nu_j)}{4 \sum_{j=1}^{N} (\nu_j - \mu_j)} \tag{3.9}$$

in view of the equality

$$2 \sum_{i<j} \mu_i \mu_j = \left(\sum_{j=1}^{N} \mu_j\right)^2 - \sum_{j=1}^{N} \mu_j^2.$$

Consider the function $\det W$, which is a $(2N+2)$th-order polynomial in λ with constant coefficients of the x-flow and t_m-flow,

$$-\det \mathrm{W} = f^2 + gh = \tfrac{1}{4} \prod_{j=1}^{2N+2} (\lambda - \lambda_j) = \tfrac{1}{4} R(\lambda) \qquad \lambda_{2N+2} = 0. \tag{3.10}$$

Substituting (2.13) into (3.10) and comparing the coefficients of λ^{2N+1} and λ^{2N} yields

$$2f_{-1}f_0 + g_{-1}h_{-1} = -\tfrac{1}{4} \sum_{j=1}^{2N+2} \lambda_j$$
$$2f_{-1}f_1 + f_0^2 + g_{-1}h_0 + g_0 h_{-1} = \tfrac{1}{4} \sum_{i<j} \lambda_i \lambda_j$$

which together with (3.1) lead to

$$\alpha_1 = -\tfrac{1}{2} \sum_{j=1}^{2N+2} \lambda_j \qquad \alpha_2 = \tfrac{1}{2} \sum_{i<j} \lambda_i \lambda_j - \tfrac{1}{8} \left(\sum_{j=1}^{2N+2} \lambda_j\right)^2. \tag{3.11}$$

From (3.10), we see that

$$f|_{\lambda=\mu_k} = \tfrac{1}{2}\sqrt{R(\mu_k)} \qquad f|_{\lambda=\nu_k} = \tfrac{1}{2}\sqrt{R(\nu_k)}. \tag{3.12}$$

Noticing (3.2) and (2.11), we obtain

$$g_x|_{\lambda=\mu_k} = -u\mu_k\mu_{kx}\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i) = -2u\mu_k f|_{\lambda=\mu_k}$$
$$h_x|_{\lambda=\nu_k} = -v\nu_{kx}\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i) = 2vf|_{\lambda=\nu_k} \qquad 1\leqslant k\leqslant N \tag{3.13}$$

which together with (3.12) gives

$$\mu_{kx} = \frac{\sqrt{R(\mu_k)}}{\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)} \qquad \nu_{kx} = -\frac{\sqrt{R(\nu_k)}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)} \qquad 1\leqslant k\leqslant N. \tag{3.14}$$

From (2.5), (3.5) and (3.4), we have

$$V_{12}^{(1)}|_{\lambda=\mu_k} = u\mu_k\left(\mu_k-\sum_{j=1}^{N}\mu_j-\alpha_1\right) \qquad V_{21}^{(1)}|_{\lambda=\nu_k} = v\left(\nu_k-\sum_{j=1}^{N}\nu_j-\alpha_1\right) \tag{3.15}$$

$$V_{12}^{(2)}|_{\lambda=\mu_k} = u\mu_k\left[\mu_k^2-\mu_k\sum_{j=1}^{N}\mu_j+\sum_{i<j}\mu_i\mu_j+\alpha_1\left(\sum_{j=1}^{N}\mu_j-\mu_k\right)+\alpha_1^2-\alpha_2\right]$$
$$V_{21}^{(2)}|_{\lambda=\nu_k} = v\left[\nu_k^2-\nu_k\sum_{j=1}^{N}\nu_j+\sum_{i<j}\nu_i\nu_j+\alpha_1\left(\sum_{j=1}^{N}\nu_j-\nu_k\right)+\alpha_1^2-\alpha_2\right]. \tag{3.16}$$

In a way similar to the calculation of (3.14), we arrive at

$$\mu_{kt_m} = \frac{\sqrt{R(\mu_k)}u^{-1}V_{12}^{(m)}|_{\lambda=\mu_k}}{\mu_k\prod_{i=1,i\neq k}^{N}(\mu_k-\mu_i)}$$
$$\nu_{kt_m} = -\frac{\sqrt{R(\nu_k)}v^{-1}V_{21}^{(m)}|_{\lambda=\nu_k}}{\prod_{i=1,i\neq k}^{N}(\nu_k-\nu_i)} \qquad 1\leqslant k\leqslant N, 1\leqslant m\leqslant 2. \tag{3.17}$$

Therefore, if the $(2N+2)$ distinct parameters $\lambda_1,\ldots,\lambda_{2N+2}(\lambda_{2N+2}=0)$ are given, and let $\mu_k(x,t_m)$ and $\nu_k(x,t_m)$ be distinct solutions of ordinary differential equations (3.14) and (3.17), then (u,v) determined by (3.5) and (3.6) is a solution of equation (1.5) with $m=1$ or the higher-order equation (1.6) with $m=2$. This means that q by (3.9) is a solution of the mKP equation (1.1).

4. Quasi-periodic solutions

In this section, we shall give the quasi-periodic solutions of the mKP equation (1.1). To this end we first introduce the Riemann surface Γ of the hyperelliptic curve $\zeta^2=R(\lambda)$, $R(\lambda)=\prod_{j=1}^{2N+2}(\lambda-\lambda_j)$, of genus N. On Γ there are two infinite points ∞_1 and ∞_2, which are not branch points of Γ. Equip Γ with a canonical basis of cycles: $a_1,\ldots,a_N;b_1,\ldots,b_N$, which are independent and have intersection numbers as follows:

$$a_i\circ a_j=0 \qquad b_i\circ b_j=0 \qquad a_i\circ b_j=\delta_{ij}.$$

For the present, we will choose as our basis the following set:

$$\varpi_l = \frac{\lambda^{l-1}\,\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \qquad 1\leqslant l\leqslant N$$

which are N linearly independent homomorphic differentials on Γ. By using the cycles a_j and b_j, the period matrices A and B can be constructed from

$$A_{ij} = \int_{a_j} \varpi_i \qquad B_{ij} = \int_{b_j} \varpi_i.$$

It is possible to show that the matrices A and B are invertible [17, 18]. Now we define the matrices C and τ by $C = A^{-1}, \tau = A^{-1}B$. The matrix τ can be shown to be symmetric ($\tau_{ij} = \tau_{ji}$) and it has a positive-definite imaginary part (Im $\tau > 0$). If we normalize ϖ_l into the new basis ω_j,

$$\omega_j = \sum_{l=1}^{N} C_{jl}\varpi_l \qquad 1 \leqslant j \leqslant N$$

then we have

$$\int_{a_i} \omega_j = \sum_{l=1}^{N} C_{jl} \int_{a_i} \varpi_l = \sum_{l=1}^{N} C_{jl}A_{li} = \delta_{ji}$$

and

$$\int_{b_i} \omega_j = \tau_{ji}.$$

Now we introduce Abel–Jacobi coordinates as follows:

$$\rho_j^{(1)}(x, y, t) = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{p_0}^{\mu_k} \frac{\lambda^{l-1}\,\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \qquad 1 \leqslant j \leqslant N \tag{4.1}$$

$$\rho_j^{(2)}(x, y, t) = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(x,y,t)} \omega_j = \sum_{k=1}^{N}\sum_{l=1}^{N} C_{jl} \int_{p_0}^{\nu_k} \frac{\lambda^{l-1}\,\mathrm{d}\lambda}{\sqrt{R(\lambda)}} \qquad 1 \leqslant j \leqslant N \tag{4.2}$$

where p_0 is chosen a base point on Γ. From the first expression of (3.14), we obtain

$$\partial_x \rho_j^{(1)} = \sum_{l=1}^{N}\sum_{k=1}^{N} C_{jl} \frac{\mu_k^{l-1}\mu_{kx}}{\sqrt{R(\mu_k)}} = \sum_{l=1}^{N}\sum_{k=1}^{N} \frac{\mu_k^{l-1}C_{jl}}{\prod_{i\neq k}^{N}(\mu_k - \mu_i)}$$

which implies

$$\partial_x \rho_j^{(1)} = C_{jN} = \Omega_j^{(0)} \qquad 1 \leqslant j \leqslant N \tag{4.3}$$

with the help of the following equality:

$$\sum_{k=1}^{N} \frac{\mu_k^{l-1}}{\prod_{i\neq k}^{N}(\mu_k - \mu_i)} = \delta_{lN} \qquad 1 \leqslant l \leqslant N. \tag{4.4}$$

In a similar way, we obtain from (3.16) that

$$\partial_y \rho_j^{(1)} = \Omega_j^{(1)} \qquad \partial_t \rho_j^{(1)} = \Omega_j^{(2)} \qquad 1 \leqslant j \leqslant N \tag{4.5}$$

$$\partial_x \rho_j^{(2)} = -\Omega_j^{(0)} \qquad \partial_y \rho_j^{(2)} = -\Omega_j^{(1)} \qquad \partial_t \rho_j^{(2)} = -\Omega_j^{(2)} \qquad 1 \leqslant j \leqslant N \tag{4.6}$$

where

$$\Omega_j^{(1)} = C_{j,N-1} - \alpha_1 C_{jN} \qquad \Omega_j^{(2)} = C_{j,N-2} - \alpha_1 C_{j,N-1} + (\alpha_1^2 - \alpha_2)C_{jN}.$$

On the basis of these results we obtain the following:

$$\rho_j^{(1)}(x, y, t) = \Omega_j^{(0)}x + \Omega_j^{(1)}y + \Omega_j^{(2)}t + \gamma_j^{(1)} \qquad 1 \leqslant j \leqslant N \tag{4.7}$$

$$\rho_j^{(2)}(x, y, t) = -\Omega_j^{(0)}x - \Omega_j^{(1)}y - \Omega_j^{(2)}t + \gamma_j^{(2)} \qquad 1 \leqslant j \leqslant N \tag{4.8}$$

where $\gamma_j^{(m)}$'s ($m = 1, 2$) are constants,

$$\gamma_j^{(1)} = \sum_{k=1}^{N} \int_{p_0}^{\mu_k(0,0,0)} \omega_j \qquad \gamma_j^{(2)} = \sum_{k=1}^{N} \int_{p_0}^{\nu_k(0,0,0)} \omega_j.$$

Let $\mathcal{T}$ be the lattice set generated by the $2N$ vectors $\{\delta_j, \tau_j\}$, where $\delta_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{N-j})^T$ and $\tau_j = \tau\delta_j$. The complex torus $\mathcal{J} = \mathbb{C}^N/\mathcal{T}$ is called the Jacobian variety of Γ. An Abel map $\mathcal{A} : \mathrm{Div}(\Gamma) \to \mathcal{J}$ is defined as

$$\mathcal{A}(p) = \int_{p_0}^{p} \omega \qquad \omega = (\omega_1, \ldots, \omega_N)^T$$

with the natural linear extension to the factor group $\mathrm{Div}(\Gamma)$

$$\mathcal{A}\left(\sum n_k p_k\right) = \sum n_k \mathcal{A}(p_k).$$

Consider two special divisors $\sum_{j=1}^{N} p_k^{(m)}$ ($m = 1, 2$) and

$$\mathcal{A}\left(\sum_{k=1}^{N} p_k^{(m)}\right) = \sum_{k=1}^{N} \mathcal{A}(p_k^{(m)}) = \sum_{k=1}^{N} \int_{p_0}^{p_k^{(m)}} \omega = \rho^{(m)}$$

with $p_k^{(1)} = (\mu_k, \zeta(\mu_k))$ and $p_k^{(2)} = (\nu_k, \zeta(\nu_k))$, whose components are

$$\sum_{k=1}^{N} \int_{p_0}^{p_k^{(m)}} \omega_j = \rho_j^{(m)} \qquad 1 \leqslant j \leqslant N \quad m = 1, 2.$$

According to the Riemann theorem [17, 18], there exists a constant vector $M^{(m)} = (M_1^{(m)}, \ldots, M_N^{(m)})^T \in \mathbb{C}^N$ such that the function

$$F^{(m)}(\lambda) = \theta(\mathcal{A}(p) - \rho^{(m)} - M^{(m)}) \qquad m = 1, 2 \tag{4.9}$$

has exactly N zeros at $\mu_1, \ldots, \mu_N$ (for $m = 1$) or $\nu, \ldots, \nu_N$ (for $m = 2$), where $p = (\lambda, \zeta)$ and θ is the Riemann theta function defined by

$$\theta(\xi|\tau) = \sum_{z \in \mathbb{Z}^N} \exp(\pi\sqrt{-1}\langle \tau z, z\rangle + 2\pi\sqrt{-1}\langle \xi, z\rangle)$$

in which $\xi = (\xi_1, \ldots, \xi_N)^T \in \mathbb{C}^N$, $\langle \xi, z\rangle = \sum_{j=1}^{N} \xi_j z_j$.

To make the function single valued, the surface Γ is cut along all a_k, b_k to form a simple connected region, whose boundary is denoted by γ. Notice the fact that the integrals [17, 18]

$$\frac{1}{2\pi\sqrt{-1}} \int_{\gamma} \lambda^k \,\mathrm{d} \ln F^{(m)}(\lambda) = I_k(\Gamma)$$

are constants independent of $\rho^{(m)}$ with

$$I_k(\Gamma) = \sum_{j=1}^{N} \int_{a_j} \lambda^k \omega_j.$$

By the residue theorem, we have

$$\begin{aligned} I_k(\Gamma) &= \sum_{l=1}^{N} \mu_l^k + \sum_{s=1}^{2} \mathrm{Res}_{\lambda=\infty_s} \lambda^k \,\mathrm{d} \ln F^{(1)}(\lambda) \\ I_k(\Gamma) &= \sum_{l=1}^{N} \nu_l^k + \sum_{s=1}^{2} \mathrm{Res}_{\lambda=\infty_s} \lambda^k \,\mathrm{d} \ln F^{(2)}(\lambda). \end{aligned} \tag{4.10}$$

Here we need only compute the residues in (4.10) for $k = 1, 2$. In a way similar to calculations in [9, 14, 15], we arrive at

$$\begin{aligned}&\mathrm{Res}_{\lambda=\infty_s}\,\lambda\,\mathrm{d}\ln F^{(m)}(\lambda) = (-1)^{s+m}\partial\ln\theta_s^{(m)}\\&\mathrm{Res}_{\lambda=\infty_s}\,\lambda^2\,\mathrm{d}\ln F^{(m)}(\lambda) = (-1)^{s+m}\partial_y\ln\theta_s^{(m)} + \partial^2\ln\theta_s^{(m)} \qquad 1\leqslant m\leqslant 2, 1\leqslant s\leqslant 2\end{aligned} \tag{4.11}$$

where

$$\theta_s^{(1)} = \theta(\Omega^{(0)}x + \Omega^{(1)}y + \Omega^{(2)}t + \Upsilon^{(s)}) \qquad \theta_s^{(2)} = \theta(-\Omega^{(0)}x - \Omega^{(1)}y - \Omega^{(2)}t + \Lambda^{(s)})$$

with

$$\Omega^{(i)} = (\Omega_1^{(i)},\ldots,\Omega_N^{(i)})^T \qquad \Upsilon^{(s)} = (\Upsilon_1^{(s)},\ldots,\Upsilon_N^{(s)})^T \qquad \Lambda^{(s)} = (\Lambda_1^{(s)},\ldots,\Lambda_N^{(s)})^T$$

$$\Upsilon_j^{(s)} = \gamma_j^{(1)} + M_j^{(1)} + \int_{\infty_s}^{p_0}\omega_j \qquad \Lambda_j^{(s)} = \gamma_j^{(2)} + M_j^{(2)} + \int_{\infty_s}^{p_0}\omega_j$$

$$0\leqslant i\leqslant 2 \quad 1\leqslant j\leqslant N.$$

Equations (4.10) and (4.11) implies

$$\begin{aligned}&\sum_{l=1}^{N}\mu_l(x,y,t) = I_1(\Gamma) + \partial\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}}\\&\sum_{l=1}^{N}\nu_l(x,y,t) = I_1(\Gamma) + \partial\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}}\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\sum_{l=1}^{N}\mu_l^2(x,y,t) = I_2(\Gamma) + \partial_y\ln\frac{\theta_2^{(1)}}{\theta_1^{(1)}} - \partial^2\ln\theta_1^{(1)}\theta_2^{(1)}\\&\sum_{l=1}^{N}\nu_l^2(x,y,t) = I_2(\Gamma) + \partial_y\ln\frac{\theta_1^{(2)}}{\theta_2^{(2)}} - \partial^2\ln\theta_1^{(2)}\theta_2^{(2)}.\end{aligned} \tag{4.13}$$

Substituting (4.12) and (4.13) into (3.9), we obtain the quasi-periodic solutions of the mKP equation (1.1)

$$\begin{aligned}q(x,y,t) &= \tfrac{1}{2}(\alpha_1 + I_1) + \tfrac{1}{4}\partial\ln\frac{\theta_2^{(1)}\theta_1^{(2)}}{\theta_1^{(1)}\theta_2^{(2)}}\\&\quad+\frac{\partial_y\ln\big(\theta_1^{(1)}\theta_1^{(2)}/\theta_2^{(1)}\theta_2^{(2)}\big) + \partial^2\ln\big((\theta_2^{(1)})^3\theta_1^{(2)}/\theta_1^{(1)}(\theta_2^{(2)})^3\big)}{4\partial\ln\big(\theta_1^{(1)}\theta_1^{(2)}/\theta_2^{(1)}\theta_2^{(2)}\big)}.\end{aligned} \tag{4.14}$$

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